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An Intersection Theorem for Supermatroids

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We generalize the matroid intersection theorem to distributive supermatroids, a structure that extends the matroid to the partially ordered ground set. Distributive supermatroids are special cases of both supermatroids and greedoids, and they generalize polymatroids. This is the first good characterization proved for the intersection problem of an independence system where the ground set is partially ordered. The characterization given has a more complex structure than the matroid (or polymatroid) intersection theorem. © 1990 Academic Press, Inc.

1. Introduction

The concept of a matroid is an important unifying concept in combinatorics. There have been several generalizations suggested. The most important ones seem to be polymatroids [Edm70], supermatroids [DIW72], F-geometries [Fai80], and greedoids [KL84]. Among these, greedoids proved to be the richest in interesting combinatorial examples that are not matroids.

Another important concept in combinatorics is the partially ordered set. All of the above generalizations, except polymatroids, can be considered as introducing ordered sets in one of the definitions of a matroid.

Polymatroids were defined to generalize properties of matroids relevant from a combinatorial optimization point of view (the optimality of the greedy algorithm and the matroid intersection theorem). The other three structures, which all contain polymatroids as special cases, proved to be quite successful in generalizing the structural properties of matroids. (See [KL83] for structural results for greedoids.) So far no generalization of the matroid intersection theorem, the most important theorem from an optimization point of view, could be proved for any of the more general structures. In fact the intersection problem is already NP-hard in quite simple common special cases of these structures.

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In this paper we give a generalization of the matroid intersection theorem to a special class of supermatroids, distributive supermatroids, defined on the same partially ordered set. This special case was suggested to us by Ulrich Faigle.

In fact distributive supermatroids are not only special cases of supermatroids, but are also special cases of both greedoids and F-geometries. In view of the richness of greedoids in interesting examples it would be very important to extend this intersection theorem to a larger class of greedoids covering more of the interesting examples. However, this is the first generalization of the matroid intersection theorem to a setting where the underlying set is partially ordered; we consider this to be the main contribution of this paper.

We give two forms of the intersection theorem. One is a direct generalization of the matroid intersection theorem, but it is not a good characterization. The other min–max theorem is more complicated, but it provides a good characterization.

The intersection problem for greedoids and distributive supermatroids can be formulated as follows: Both greedoids and distributive supermatroids can be defined as independence systems. The intersection problem is to find the maximum cardinality common independent set in two such independence systems. We shall show that the intersection problem for distributive supermatroids (given by independence oracles) is not solvable in polynomial time (furthermore, it has NP-complete special cases that can be defined without an oracle). Therefore, in general, one cannot hope for a good characterization. We consider a special case, when the two supermatroids are defined on the same partially ordered set, where a good characterization is possible.

The paper is structured as follows: After the Introduction, the section Definitions and Preliminaries gives the definition of a distributive supermatroid and presents some examples and basic notation. In the section The Intersection Problem we give more definitions and prove basic lemmas in order to be able to state the two forms of our main theorem: Theorem 3 (the usual form) and Theorem 8 (the good characterization). We are going to prove the trivial max $\leqslant$ min direction of Theorem 8 and show that Theorem 8 implies Theorem 3. The main content of the paper, the nontrivial direction of Theorem 8 will be proved in a separate section entitled Proof of the Intersection Theorem.

2. Definitions and Preliminaries

For a set X and an element x we shall use $X+x$ to denote $X \cup \{x\}$. Let P be a partially ordered set on S, and X be a subset of S. For two elements

x and y of P the fact that x is above y in the partial order P will be denoted by $x \geqslant y$. Let $[X] = \{y \in P : \exists x \in X \text{ with } x \geqslant y\}$ be the ideal generated by X and $[X]^d = \{y \in P : \exists x \in X \text{ with } x \leqslant y\}$ be the dual ideal generated by X; if $X = \{x\}$ we shall use $[x]$ and $[x]^d$ to denote $[\{x\}]$ and $[\{x\}]^d$, respectively.

Let P denote a partially ordered set on the ground set S. A subset $\mathscr{F}$ of the ideals of P form the independent sets of a (distributive) *supermatroid* if the following three conditions hold:

- (S1) $\varnothing \in \mathscr{F}$,
- (S2) if $Y \subseteq X$ is an ideal and $X \in \mathscr{F}$ then $Y \in \mathscr{F}$,
- (S3) $|Y| < |X|$ and $X, Y \in \mathscr{F}$ implies that $\exists x \in X \setminus Y$ such that $Y + x \in \mathscr{F}$.

Note that the same axioms are required as the independence axioms of matroids, except that all independent sets are supposed to be ideals of the partially ordered set P.

The above definition makes it apparent that distributive supermatroids are greedoids. Supermatroids were defined by Dunstan, Ingleton, and Welsh [DIW72] as a set of elements of a lattice satisfying certain conditions. The above definition is an alternative way to describe the conditions required for $\mathscr{F}$ to be a supermatroid, when considered as a set of elements in the lattice of ideals of the partially ordered set P. The name "distributive supermatroid" is justified by the fact that distributive lattices are exactly the lattices of ideals of partially ordered sets. In this paper we shall deal with distributive supermatroids only, and we shall refer to the above definition by the term supermatroid. Let us mention some examples of (distributive) supermatroids:

(E1) Let P be the partial ordere on a ground set S where no two elements are compatible. Supermatroids on P are exactly the matroids on the ground set S.

(E2) Let P consist of disjoint chains whose elements are incompatible. Supermatroids on P correspond to polymatroids on the set of chains of P. Indeed an ideal of P can be described by an integer vector on the set of chains of P (indicating how many elements of the chain are contained in the ideal). A set of ideals on P forms the independent sets of a supermatroid if and only if the corresponding vectors are the integer vectors of a polymatroid on the set of chains of P.

(E3) One can define the uniform supermatroid on any partially ordered set: Given a partially ordered set P and an integer k, those ideals of P which have at most k elements form a supermatroid on P.

(E4) We can also define the analog of the transversal matroid: Given a partially ordered set P and a set of ideals $\mathscr{A} = \{A_i : i \in I\}$ of P, those sets of partial representatives of the family A, which themselves are ideals of P, form the independent sets of a supermatroid.

Recall from [DIW72] the generalization of the notion of contraction and deletion for supermatroids. These notions can be defined analogously to the notion of deletion and contraction in matroids. For a supermatroid $\mathscr{F}$ defined on the partially ordered set P and a dual ideal $[X]^{\mathrm{d}}$ the *deletion* of $[X]^{\mathrm{d}}$ results in a supermatroid $\mathscr{F} \backslash [X]^{d} = \{A \in \mathscr{F} : \text{such that } A \cap [X]^{\mathrm{d}} = \varnothing\}$ that is a supermatroid on the partially ordered set $P \backslash [X]^{\mathrm{d}}$. For an independent ideal A of P the *contraction* of A results in a supermatroid $\mathscr{F}/A = \{B \backslash A : A \subseteq B \in \mathscr{F}\}$ which is a supermatroid on the partially ordered set $P \backslash A$. For an element p in P we shall use $\mathscr{F}/p$ to denote $\mathscr{F}/\{p\}$.

3. The Intersection Problem

First we show that the intersection problem for two (distributive) supermatroids defined on two, possibly different, partially ordered sets cannot be solved in polynomial time. Given a matroid $\mathscr{M}$ on the ground set $S = \{1, 2, ..., 2n\}$ and an integer k the *matroid matching problem* is to decide whether there exists a *matroid* matching of size k, i.e., an independent set I in $\mathscr{M}$ of size $2k$ that for all i contains either both i and $n+i$ or neither of them. For matroids given by independence oracles the matroid matching problem is not solvable in polynomial time. Furthermore there are matroids that can be defined without oracles where the matroid matching problem is NP-complete. (See in [LP86].)

Theorem 1. *The problem of finding the maximum common independent set of two supermatroids defined on different partially ordered sets contains the matroid matching problem as a special case.*

Proof. Consider an instance of the matroid matching problem with a matroid $\mathscr{M}$ on the ground set $S = \{1, 2, ..., 2n\}$ and an integer k. We reduce this problem to the supermatroid intersection problem. The two supermatroids will be defined on the set $S' = \{1, ..., 3n\}$. The first supermatroid $\mathscr{F}_1$ is defined on the partially ordered set P_1, where no two elements of S' are compatible. The supermatroid $\mathscr{F}_1$ is the matroid $\mathscr{F}_1 = \mathscr{M} \oplus U_k$, i.e., the direct sum of the matroid $\mathscr{M}$ with the uniform matroid U_k of rank k on the elements $\{2n+1, ..., 3n\}$. The other supermatroid is defined on the patially ordered set P_2 on S', where the order-relations in P_2 are $i \geqslant n+i \geqslant 2n+i$ for all $1 \leqslant i \leqslant n$. The supermatroid $\mathscr{F}_2$ is the uniform supermatroid of rank $3k$ on P_2. These two supermatroids have a common independent set of size $3k$ if and only if the matroid $\mathscr{M}$ has a matching of size k. ∎

Next we turn to the intersection problem for two supermatroids defined on the same partially ordered set P. We want to extend the natural notion of the rank of an ideal to sets that are not ideals. There are two alternative ways to do this. Conforming with the usual terminology in the theory of greedoids, for a supermatroid $\mathscr{F}$ we define the *rank* of a set $X \subseteq S$ as

$$r(X) = \max(|I| : I \in \mathscr{F} \text{ and } I \subseteq X); \tag{1}$$

and the *basis-rank* of X as

$$\beta(X) = \max(|I \cap X| : I \in \mathscr{F}). \tag{2}$$

Note that the rank of any set can be computed with the greedy algorithm just as in the case of matroids, whereas no polynomial time method exists to calculate the basis-rank of a set. (The latter fact can be proved similarly to Theorem 1.)

PROPOSITION 2. *Given a partially ordered set P and a supermatroid $\mathscr{F}$ defined on P by an independence oracle, the* rank *$r(X)$ of a set X can be computed in polynomial time by the greedy algorithm.*

For two supermatroids $\mathscr{F}_1$ and $\mathscr{F}_2$ we shall use the notation r_1, r_2, β_1, and β_2 for their respective rank and basis-rank functions. The following is the usual form of the intersection theorem.

THEOREM 3. *Let P be a partially ordered set on the ground set S. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two supermatroids defined on the partially ordered set P*:

$$\max(|I| \textit{ for } I \in \mathscr{F}_1 \cap \mathscr{F}_2) = \min(\beta_1(X) + \beta_2(S \setminus X) \textit{ for } X \subseteq S). \tag{3}$$

Note that this formulation of the theorem, though it is clearly analogous to the matroid intersection theorem, does not give a good characterization. It is not clear how one would compute the basis rank of the sets X and $S \setminus X$ on the right-hand side. The main content of the good characterization (Theorem 8) is that it suffices to take X in a special form in which the basis-rank of both X and $S \setminus X$ can be computed. We establish two special cases where the basis-rank of a set can be computed efficiently.

Let $\mathscr{F}$ be a supermatroid on a partially ordered set P and let X be an ideal of P. An element x of X is an *isthmus* in X if x is contained in all maximal independent subsets of X. We shall use the notation $I(X)$ for the isthmuses of X. For any ideal X the set of istmuses $I(X)$ is also an ideal.

LEMMA 4. *For any ideal $X \in \mathscr{F}$ the basis-rank of $X \setminus I(X)$ is* $r(X) - |I(X)|$.

Next we establish a technical lemma that states that a property similar to (S3) is also true for the basis-rank function instead of the rank function.

LEMMA 5. *If $I \subseteq X \subseteq S$ and $I \in \mathscr{F}$ then there exists $K \in \mathscr{F}$ such that $I \subseteq K$ and $|K \cap X| = \beta(X)$.*

Proof. Let J be an ideal in $\mathscr{F}$ such that $|X \cap J| = \beta(X)$. By applying (S3) repeatedly we find an ideal $K \in \mathscr{F}$, such that $I \subseteq K \subseteq I \cup J$ and $|K| = |J|$. Now $|K \cap X| \geqslant |K| - |(I \cup J) \backslash X| = |J| - |J \backslash X| = |J \cap X|$. ∎

For an ideal X, an element $y \in S$ is called *dependent* on X if either $y \in X$ or there exists an $x \in X$ that is not an isthmus and $x \leqslant y$. The set of all dependents of an ideal X will be denoted by $D(X)$. The name dependent is suggested by the following lemma:

LEMMA 6. *For an ideal X we have $\beta(D(X)) = r(X)$.*

Proof. Let y be dependent on the ideal X. We first prove the special case that $r(X) = \beta(X + y)$. By definition there exists an element x in X such that x is not an isthmus and $x \leqslant y$. Let I be a maximal independent set in X that does not contain x. Using Lemma 5 for the sets $X + y$ (in place of X) and I, we get an independent set K such that $I \subseteq K$ and $|K \cap (X + y)| = \beta(X + y)$. However K is independent and it contains the set I, a maximum independent subset of X, thus $X \cap K = I$. Therefore $x \notin K$, and since K is an ideal $y \notin K$ either. This implies $\beta(X + y) = |K \cap (X + y)| = |I| = r(X)$.

The lemma is proved by contradiction. Let I be a maximal independent set in X. Applying Lemma 5 to $D(X)$ and I, we get an indeendent set K that contains I and $|K \cap D(X)| = \beta(D(X))$. Now if $\beta(D(X)) > r(X)$ then there exists an independent set $L \subseteq K$ such that $I \subseteq L$ and $|L \cap D(X)| = r(X) + 1$. This contradicts the special case proved above for $\{y\} = L \cap X \backslash I$. ∎

The following two observations concerning the monotonicity of the functions $I(\cdot)$ and $D(\cdot)$ will prove useful.

LEMMA 7. *If X and Y are ideals in P and $X \subseteq Y$, then $I(Y) \cap X \subseteq I(X)$ and $D(X) \subseteq D(Y)$.*

Given two supermatroids $\mathscr{F}_1$ and $\mathscr{F}_2$ we shall use the notation $D_1(X)$, $D_2(X)$, $I_1(X)$, and $I_2(X)$ for the set of dependents and isthmuses of a set X in the two supermatroids, respectively.

THEOREM 8. *Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two supermatroids defined on the same partially ordered set P on a ground set S; then*

$$\max(|I| \textit{ for } I \in \mathscr{F}_1 \cap \mathscr{F}_2) = \min(r_1(X_1) + r_2(X_2) - |X_1 \cap X_2|),$$

where the minimum is taken over sets X_1 and X_2 that are ideals of P such that

$$I_i(X_i) = X_1 \cap X_2 \qquad \text{and} \qquad [X_{3-i} \backslash X_i]^{\mathrm{d}} = S \backslash X_i \qquad \text{for} \quad i = 1, 2. \quad (4)$$

Proof of max $\leqslant$ min. Let I be a common independent set and let X_1 and X_2 be ideals satisfying the conditions required on the right-hand side. By Lemmas 4 and 6, $\beta_2(X_2 \backslash X_1) = r_2(X_2) - |X_1 \cap X_2|$ and $r_1(X_1) = \beta_1(D_1(X_1)) = \beta_1((S \backslash X_2) \cup X_1))$. Consequently

$$r_1(X_1) + r_2(X_2) - |X_1 \cap X_2| = \beta_1(S \backslash Y) + \beta_2(Y) \leqslant |I \backslash Y| + |I \cap Y| = |I|$$

for $Y = X_2 \backslash X_1$. ∎

The min $\leqslant$ max direction will be proved in the next section. Here we derive Theorem 3.

Proof of Theorem 3. The max $\leqslant$ min direction is trivial. To prove the other direction, one has to exhibit a set Y such that $\beta_1(S \backslash Y)) + \beta_2(Y) = \max(|I| \text{ for } I \in \mathscr{F}_1 \cap \mathscr{F}_2)$. Let X_1 and X_2 be the two ideals where the minimum in Theorem 8 is attained by the theorem that minimum is equal to $\max(|I| \text{ for } I \in \mathscr{F}_1 \cap \mathscr{F}_2)$. As was shown in the proof of the max $\leqslant$ min direction of Theorem 8 for $Y = X_2 \backslash X_1$,

$$\beta_1(S \backslash Y)) + \beta_2(Y) = r_1(X_1) + r_2(X_2) - |X_1 \cap X_2|. \quad ∎$$

4. Proof of the Intersection Theorem

The next lemma proves that some of the conditions in Theorem 8 can be relaxed.

LEMMA 9. *If there are ideals A and B such that*

$$A \cap B \subseteq I_1(A), I_2(B) \subseteq A \cap B \qquad \text{and} \qquad [B \backslash A]^{\mathrm{d}} = S \backslash A \qquad (5)$$

then there are ideals X_1 and X_2 satisfying (4) *such that*

$$r_1(X_1) + r_2(X_2) - |X_1 \cap X_2| \leqslant r_1(A) + r_2(B) - |A \cap B|.$$

Proof. Choose the ideals A and B satisfying (5) such that $r_1(A) + r_2(B) - |A \cap B|$ is minimum, A is minimal, and B is maximal. Now we claim that $X_1 = A$ and $X_2 = B$ will satisfy (4).

First we show $I_1(A) = A \cap B$. The right-hand side is contained on the left-hand side by assumption. To prove the reverse containment consider the sets $A' = A$ and $B' = B \cup I_1(A)$. These sets also satisfy (5). Furthermore $r_1(A') + r_2(B') - |A' \cap B'| \leqslant r_1(A) + r_2(B) - |A \cap B|$ and $A' = A$. Hence the minimal choice of $r_1(A)+r_2(B)-|A\cap B|$ and the maximal choice of B implies $B'=B$.

Next we prove that $[A\backslash B]^{\mathrm{d}}=S\backslash B$. The $\subseteq$ direction is trivial. To see the other direction consider the sets $A'=A$ and $B'=B\cup(S\backslash[A\backslash B]^{\mathrm{d}})$. A' and B' satisfy (5). The assumptions $[B\backslash A]^{\mathrm{d}}=S\backslash A$ and $I_2(B)\subseteq A\cap B$ and Lemma 6 imply that $r_2(B)=r_2(B')$. The maximal choice of B implies that $B'=B$.

Finally we prove that $I_2(B)=A\cap B$. Here the left-hand side is contained on the right-hand side by assumption. The reverse containment is proved by contradiction. Let x be a maximal element in $(A\cap B)\backslash I_2(B)$. Consider the sets $A'=A\backslash[x]^{\mathrm{d}}$ and $B'=B$. These sets satisfy (5). The element x is in $I_1(A)$, therefore, $r_1(A')\leqslant r_1(A)-1$ and $r_1(A')+r_2(B')-|A'\cap B'|\leqslant r_1(A)-1+r_2(B)-(|A\cap B|-1)$, contradicting the minimal choice of A. ∎

We need a slightly stronger version of the usual submodularity of the rank-function of a supermatroid on ideals. Let r be the rank-function of a supermatroid $\mathscr{F}$ on the partially ordered set P. For an ideal A of P let $I(A)$ denote the set of isthmuses and $D(A)$ denote the set of dependents of A.

LEMMA 10. *For any two ideals A and B of P*

$$r(A)+r(B)\geqslant r(A\cap B)+r(A\cup B)+|(I(A)\cap D(B))\backslash B|. \tag{6}$$

Proof. Let I be a maximal independent set in $A\cap B$, let J be a maximal independent set in B that contains I, and let K be a maximal independent set in $A\cup B$ that contains J. By (S3) we know that $|I|=r(A\cup B)$, $|J|=r(B)$, and $|K|=r(A\cap B)$. Observe that by Lemma 6 the set $K\backslash J$ does not contain any elements from $D(B)$, consequently, $|I\cup(K\backslash J)|\leqslant r(A\backslash(D(B)\backslash B))$. Furthermore $r(A\backslash(D(B)\backslash B))\leqslant r(A)-|I(A)\cap(D(B)\backslash B)|$. This implies

$$\begin{aligned} r(A\cap B)+r(A\cup B)&=|I|+|K|=|J|+|I\cup(K\backslash J)|\\ &\leqslant r(B)+r(A\backslash(D(B)\backslash B))\\ &\leqslant r(B)+r(A)-|I(A)\cap(D(B)\backslash B)|. \quad ∎\end{aligned}$$

The proof of the main theorem (the min $\leqslant$ max direction of Theorem 8) is by induction on $|S|$. It generalizes one of the standard induction proofs of the matroid intersection theorem.

Proof of the max $\geqslant$ min *direction of Theorem* 8. The proof is by induction on $|S|$. Let $\alpha=\max(|I|:I\in\mathscr{F}_1\cap\mathscr{F}_2)$. Let p be a minimal element in P.

First we use the induction hypthesis for the supermatroids $\mathscr{F}_1/p$ and $\mathscr{F}_2/p$

on the partially ordered set $P-p$. The maximum size of a common independent set in these two supermatroids is at most $\alpha-1$. By the induction hypothesis there are ideal X_1 and X_2 containing p such that X_1-p and X_2-p satisfy (4) in $P-p$, and the sum of their respective ranks in the supermatroids $\mathscr{F}_1/p$ and $\mathscr{F}_2/p$ minus the cardinality of their intersection is at most $\alpha-1$, i.e., $(r_1(X_1)-1)+(r_2(X_2)-1)-(|X_1\cap X_2|-1)|\leqslant\alpha-1$. Equivalently

$$r_1(X_1)+r_2(X_2)-|X_1\cap X_2|\leqslant\alpha. \tag{7}$$

Let us interpret (4) in $P-p$. First $I_i(X_i)$ has to be equal either to $X_1\cap X_2$ or to $(X_1\cap X_2)-p$ (for $i=1,2$). Further $[X_3-i\backslash X_i]^{\mathrm{d}}=S\backslash X_i$ for $i=1,2$. In the case where p is an isthmus of either X_1 or X_2 (say, of X_1) then the statement follows by applying Lemma 9 (to $A=X_1$ and $B=X_2$). Assume that $I_i(X_i)=(X_1\cap X_2)-p$ for $i=1,2$.

Next consider the supermatroids $\mathscr{F}_1\backslash[p]^{\mathrm{d}}$ and $\mathscr{F}_2\backslash[p]^{\mathrm{d}}$ on $P\backslash[p]^{\mathrm{d}}$. Here the maximum common independent set has size at most α. Thus by the induction hypothesis there exist ideals Y_1 and Y_2 not containing p, satisfying the following conditions: $I_i(Y_i)=Y_1\cap Y_2$ for $i=1,2$; $(S\backslash[p]^{\mathrm{d}})\backslash Y_i\subseteq[Y_{3-i}\backslash Y_i]^{\mathrm{d}}$ for $i=1,2$; and

$$r_1(Y_1)+r_2(Y_2)-|Y_1\cap Y_2|\leqslant\alpha. \tag{8}$$

After adding inequalities (7) and (8) we shall use Lemma 10 to obtain a contradiction by "uncrossing" the sets X_1, Y_1 and X_2, Y_2. To be able to use the stronger version of the submodularity we observe the following containments:

$$(X_1\cap X_2)\backslash(Y_1\cup Y_2)-p\subseteq I_1(X_1)\cap D_1(Y_1)\backslash Y_1 \tag{9}$$

and

$$(Y_1\cap Y_2)\backslash(X_1\cap X_2)\subseteq I_2(Y_2)\cap D_2(X_2)\backslash X_2. \tag{10}$$

Now we add the inequalities (7) and (8), apply inequality (6), and use the containments (9) and (10) to get the chain of inequalities

$$\begin{aligned}
&r_1(X_1\cap Y_1)+r_2(X_2\cup Y_2)-|(X_1\cap Y_1)\cap(X_2\cup Y_2)|\\
&\qquad+r_1(X_1\cup Y_1)+r_2(X_2\cap Y_2)-|(X_1\cup Y_1)\cap(X_2\cap Y_2)|\\
&\quad\leqslant r_1(X_1)+r_1(Y_1)-|(X_1\cap X_2)\backslash(Y_1\cup Y_2)-p|\\
&\qquad+r_2(X_2)+r_2(Y_2)-|(Y_1\cap Y_2)\backslash(X_1\cup X_2)|\\
&\qquad-|(X_1\cap Y_1)\cap(X_2\cup Y_2)|-|(X_1\cup Y_1)\cap(X_2\cap Y_2)|\\
&\quad=r_1(X_1)+r_2(X_2)-|X_1\cap X_2|+r_1(Y_1)+r_2(Y_2)-|Y_1\cap Y_2|-1\\
&\quad\leqslant 2\alpha-1,
\end{aligned}$$

where the equality between the cardinalities of some sets in the last equation is valid for any four sets X_1, X_2, Y_1, and Y_2 such that $p \in (X_1 \cap X_2) \backslash (Y_1 \cup Y_2)$. Consequently either for $A = X_1 \cap Y_1$ and $B = X_2 \cup Y_2$ or for $A = X_1 \cup Y_1$ and $B = X_2 \cap Y_2$ we have $r_1(A) + r_2(B) - |A \cap B| \leqslant \alpha - 1$. We apply Lemma 9 to get a contradiction. We must show that (5) holds. By symmetry we may assume we are in the first case, i.e., $A = X_1 \cap Y_1$ and $B = X_2 \cup Y_2$.

First consider $[B \backslash A]^{\mathrm{d}} = S \backslash A$. The $\subseteq$ direction is trivial. To see the other direction observe that $S \backslash A = (S \backslash X_1) \cup (S \backslash Y_1)$. Furthermore $[B \backslash A]^{\mathrm{d}} \supseteq [X_2 \backslash X_1]^{\mathrm{d}} \cup [Y_2 \backslash Y_1]^{\mathrm{d}} \cup [p]^{\mathrm{d}}$. And finally $[Y_2 \backslash Y_1]^{\mathrm{d}} \supseteq (S \backslash Y_1) \backslash [p]^{\mathrm{d}}$ and $[X_2 \backslash X_1]^{\mathrm{d}} = S \backslash X_1$.

Next we prove $A \cap B \subseteq I_1(A)$. By Lemma 7, $I_1(A) \supseteq (I_1(X_1) \cap Y_1) \cup (I_1(Y_1) \cap X_1) = ((X_1 \cap X_2 - p) \cap Y_1) \cup (Y_1 \cap Y_2 \cap X_1) = (X_1 \cap X_2 \cap Y_1) \cup Y_1 \cap Y_2 \cap X_1) = A \cap B$.

Finally we prove that $I_2(B) \subseteq A \cap B$. Clearly $I_2(B) \subseteq B$. We must show that $I_2(B) \subseteq A$. Let x be an element from $I_2(B)$. Here we consider three subcases: $x \in X_2 \cap Y_2$, $x \in Y_2 \backslash X_2$, and $x \in X_2 \backslash Y_2$. In the first subcase, by Lemma 7, x is in $I_2(X_2) \cap I_2(Y_2) \subseteq X_1 \cap Y_1 = A$. In the second subcase, again by Lemma 7, x has to be an isthmus in Y_2, thus x belongs to $I_2(Y_2) \subseteq Y_1$. If x were in $S \backslash X_1 = [X_2 \backslash X_1]^{\mathrm{d}} \subseteq D_2(X_2)$ then x could not be an isthmus in $X_2 \cup Y_2$. Hence x is in $Y_1 \cap X_1 = A$, as required. The third subcase can be treated similarly to the second, except we have to consider the case $x = p$ separately. ∎

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